The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #1 ***Solution 3.0***

Date: September 30, 2021 Course: EE 313 Evans

Name: **Heads,** **Talking**
Last, First

- This in-person exam is scheduled to last 75 minutes.
- Open books, open notes, and open class materials, including homework assignments and solution sets and previous midterm exams and solutions.
- Calculators are allowed.
- You may use any standalone computer system, i.e. one that is not connected to a network.
- ***Please disable all wireless connections on your calculator(s) and computer system(s).***
- Please mute all computer systems.
- Please turn off all phones.
- No headphones are allowed.
- All work should be performed on the midterm exam. If more space is needed, then use the backs of the pages.
- **Fully justify your answers.** If you decide to quote text from a source, please give the quote, page number and source citation**.**

	Problem	*Point Value*	*Your score*	*Topic*
Jerry Harrison	1	22		Sampling Sinusoids
Chris Frantz	2	24		Squaring System
Tina Weymouth	3	30		AM Radio
David Byrne	4	24		Fourier Series Properties
	Total	100		

Problem 1.1 *Sampling Sinusoids*. 22 points.

SPFirst *Sec. 4-1 & 4-2*	*Lecture slides 5-5 & 5-6*
Homework Prob. 3.2	*Lecture slides 6-4 to 6-7*
	Tune-Up Tuesdays #1

Consider the sinusoidal signal $x(t) = \sin(2\,\pi\, f_0\, t + \theta)$ for continuous-time frequency f_0 in Hz.

We then sample $x(t)$ at a sampling rate f_s in Hz to produce a discrete-time signal $x[n]$.

(a) Derive the formula for $x[n]$ by sampling $x(t)$. *6 points.*

$$\boldsymbol{x[n] = x(t)|_{t=nT_s} = \sin(2\pi f_0 (nT_s) + \theta) = \sin(2\pi f_0 T_s n + \theta) = \sin\left(2\pi \frac{f_0}{f_s} n + \theta\right)}$$

$$\boldsymbol{x[n] = \sin(\hat{\omega}_0 n + \theta)}$$

(b) Based on your answer in part (a), give a formula for the discrete-time frequency $\hat{\omega}_0$ of $x[n]$ in terms of the continuous-time frequency f_0 and sampling rate f_s. Units of $\hat{\omega}_0$ are in rad/sample. *6 points.*

$$\boldsymbol{\hat{\omega}_0 = 2\pi \frac{f_0}{f_s}}$$

On the Western music scale, 392 Hz is 'G' in the 4th octave. [Ref]

(c) For continuous-time frequency f_0 = 392 Hz and sampling rate f_s = 48000 Hz,

i. What is the smallest discrete-time period in samples for $x[n]$? Why? *5 points.*

$$\boldsymbol{\hat{\omega}_0 = 2\pi \frac{f_0}{f_s} = 2\pi \frac{392 \text{ Hz}}{48000 \text{ Hz}} = 2\pi \frac{49}{6000} = 2\pi \frac{N}{L}}$$

where N and L are relatively prime integers. From Handout D *Discrete-Time Periodicity*, a discrete-time signal $x[n]$ has period N_0 if $x[n + N_0] = x[n]$ for all n.

$$\boldsymbol{x[n] = \sin(\hat{\omega}_0 n + \theta) = \sin\left(2\pi \frac{N}{L} n + \theta\right)}$$

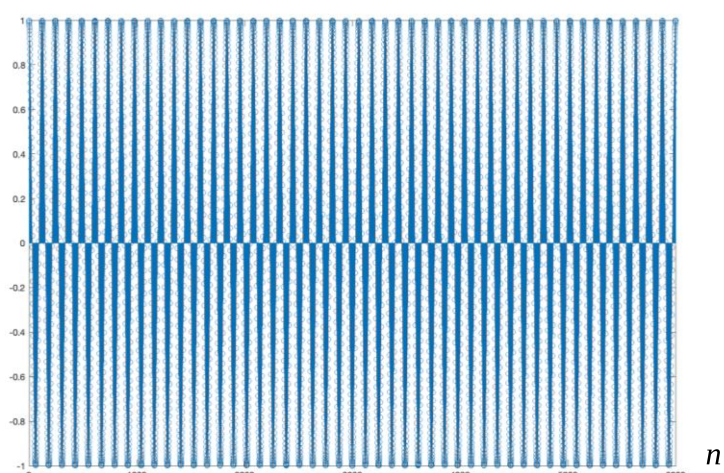
$$\boldsymbol{x[n + N_0] = \sin\left(2\pi \frac{N}{L}(n + N_0) + \theta\right) = \sin\left(2\pi \frac{N}{L} n + 2\pi \frac{N}{L} N_0 + \theta\right) = x[n]}$$

if N_0 is an integer multiple of L. The smallest period occurs when $N_0 = L$= 6000.

ii. How many continuous-time periods of $x(t)$ are in the smallest discrete-time period of $x[n]$? Why? *5 points.*

From Handout D *Discrete-Time Periodicity*, there are N continuous-time periods of $x(t)$ in the smallest discrete-time period of $x[n]$.

```
% MATLAB Code (not asked)
% Discrete-time period of 6000 samples
% contains 49 continuous-time periods
fs = 48000;
Ts = 1/fs;
f0 = 392;
wHat = 2*pi*f0/fs;
N0 = 6000;
n = 0 : N0;            % 6001 Samples
yofn = cos(wHat*n);
t = 0 : 0.01 : N0;
yoft = cos(wHat*t);
figure;
stem(n, yofn);
hold;
plot(t, yoft);
% y[n] only reaches 1 at n=0 & n=6000
find( ~(yofn - 1) )
```



Problem 1.2 *Squaring System.* 24 points.

SPFirst Sec. 3-2 to 3-5
Lecture slides 3-2 to 3-4
Lecture slides 3-7 to 3-9
Homework Prob.2.1 & 2.4
Tune-Up Tuesdays 1-3

Consider the signal $x(t) = \cos(2\,\pi\, f_1\, t) + \cos(2\,\pi\, f_2\, t)$ where $f_1 \neq f_2$.

The signal $x(t)$ is input to squaring system to produce the output $y(t) = x^2(t)$.

(a) Write $y(t)$ as a sum of cosines. What non-negative frequencies are present? Leave your answers in terms of f_1 and f_2. *9 points.*

$$y(t) = x^2(t) = (\cos(2\pi\, f_1\, t) + \cos(2\pi\, f_2\, t))^2$$

$$y(t) = x^2(t) = \cos^2(2\pi\, f_1\, t) + 2\cos(2\pi\, f_1\, t)\cos(2\pi\, f_2\, t) + \cos^2(2\pi\, f_2\, t)$$

Left term

$$\cos^2(2\,\pi\, f_1\, t) = \left(\frac{1}{2}e^{-j\,2\,\pi\, f_1\, t} + \frac{1}{2}e^{j\,2\,\pi\, f_1\, t}\right)^2 = \frac{1}{4}e^{-j\,2\,\pi\,(2\,f_1)t} + \frac{1}{2} + \frac{1}{4}e^{j\,2\,\pi\,(2\,f_1)t}$$

$$\cos^2(2\pi\, f_1\, t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi\,(2\,f_1)\,t)$$

Middle term

$$2\cos(2\pi\, f_1\, t)\cos(2\pi\, f_2\, t) = 2\left(\frac{1}{2}e^{-j\,2\,\pi\, f_1\, t} + \frac{1}{2}e^{j\,2\,\pi\, f_1\, t}\right)\left(\frac{1}{2}e^{-j\,2\,\pi\, f_2\, t} + \frac{1}{2}e^{j\,2\,\pi\, f_2\, t}\right)$$

$$= \frac{1}{2}e^{-j\,2\,\pi\,(f_1+f_2)\,t} + \frac{1}{2}e^{-j\,2\,\pi\,(f_1-f_2)\,t} + \frac{1}{2}e^{j\,2\,\pi\,(f_1-f_2)\,t} + \frac{1}{2}e^{j\,2\,\pi\,(f_1+f_2)\,t}$$

$$= \cos(2\pi\,(f_1+f_2)\,t) + \cos(2\pi\,(f_1-f_2)\,t)$$

Right term

$$\cos^2(2\pi\, f_2\, t) = \frac{1}{2} + \frac{1}{2}\cos(2\pi\,(2\,f_2)t)$$

$$1 + \frac{1}{2}\cos(2\pi\,(2\,f_1)\,t) + \cos(2\pi\,(f_1+f_2)\,t) + \cos(2\pi\,(f_1-f_2)\,t) + \frac{1}{2}\cos(2\pi\,(2\,f_2)t)$$

Non-negative frequencies: $0, 2f_1, |f_1 - f_2|, f_1 + f_2, 2f_2$

(b) For $f_1 = 110$ Hz and $f_2 = 220$ Hz, write the signal $y(t)$ using the Fourier series synthesis formula

$$y(t) = \sum_{k=-N}^{N} a_k e^{j2\pi(kf_0)t}$$

i. What is the largest possible positive value of f_0? *3 points.*

Non-negative frequencies: 0 Hz, 110 Hz, 220 Hz, 330 Hz, 440 Hz.

f_0 = gcd(110 Hz, 220 Hz, 330 Hz, 440 Hz) = 110 Hz.

ii. What is the value of N? *3 points.*

The highest harmonic frequency is 440 Hz. Hence, N = 4.

iii. Give the values of all the Fourier series coefficients a_k for k = -N, …, 0, …, N. *9 points.*

$$a_4 = a_{-4} = \frac{1}{4} \text{ and } a_3 = a_{-3} = \frac{1}{2} \text{ and } a_2 = a_{-2} = \frac{1}{4} \text{ and } a_1 = a_{-1} = \frac{1}{2} \text{ and } a_0 = 1$$

A sum of two sinusoidal signals passing through a nonlinearity such as a squaring system creates frequencies that are different from those in the original two sinusoidal signals. The intermodulation distortion in (b) occurs at 0 Hz, 330 Hz, and 440 Hz. These new frequencies are called *intermodulation distortion* because they can interfere with other frequency bands in the system. This distortion arises in data converters, power amplifiers, and other subsystems due to nonlinearities. Intermodulation distortion is a figure of merit used in audio, communications, and other systems. Intermodulation is also an intentional effect in guitar amplifiers and effects pedals to create subharmonics.

% Create a MATLAB simulation for problem 1.2. **<u>Not required for the test.</u>**
% The problem involves analysis of the output of a squaring system y(t) = x2(t).
% Let input x(t) = cos(2 p f1 t) + cos(2 p f2 t) where f1 = 110 Hz and f2 = 220 Hz.

% (a) Using a sampling rate of 8000 Hz and time from 0 to 3 seconds, write
% the code to generate a sampled version of the signal $x(t)$.

```
fs = 8000;
Ts = 1/ fs;
tmax = 3;
t = 0 : Ts : tmax;
f1 = 110;
f2 = 220;
x = cos(2*pi*f1*t) + cos(2*pi*f2*t);
```

% (b) Play sampled version of $x(t)$. Describe what you hear.
% Answer: The signal x(t) is composed of A note in the second octave (110 Hz)
% and A note in the third octave (220 Hz).
*% **On a laptop**. On many laptop speakers, the 110 Hz tone may not be audible*
% due to limitations in playing back low audible frequencies. On a laptop, the
% playback sounded like a single note with hum in the background.
*% **Audio system with a sub-woofer**. A sub-woofer plays low audible frequencies*
% down to 20 Hz. The sub-woofer is often a separate large speaker (due to the
% longer acoustic wavelengths λ for low frequencies, i.e. $\lambda = c / f$) in an audio
% system. On an audio system with a sub-woofer, the playback sounded like a
% beat frequency with hum in the background. Both notes were audible.

```
soundsc(x, fs);
pause(tmax+1);
```

% (c) Plot the spectrum of the sampled version of $x(t)$. *Principal frequencies*
% are 110 Hz and 220 Hz.
% **Using the spectrogram**. *See the second page for the plot.*

```
figure;
spectrogram(x, 512, 256, 512, fs, 'yaxis');
```

% **Using the fast Fourier transform approach from mini-project #1**.

```
fourierSeriesCoeffs = fft(x);
N = length(x);
freqResolution = fs / N;
ff = (-fs/2) : freqResolution : (fs/2)-freqResolution;
figure;
plot(ff, abs(fftshift(fourierSeriesCoeffs)));
xlabel('f');
xlim( [-1000, 1000] );
ylim( [-10, 15000] );
```

% (d) Using a sampling rate of 8000 Hz and time from 0 to 3 seconds,
% write the code to generate the sampled version of the signal $y(t)$.

```
y = x.^2;
```

% (e) Play sampled version of $y(t)$. Describe what you hear.

% Answer: The signal y(t) is composed of 'A' note in the second octave (110 Hz)
% 'A' note in the third octave (220 Hz), 'E' note in the third octave (330 Hz),
% and 'A' note in the fourth octave (440 Hz). When played back, the signal
% y(t) has a higher pitch than x(t) but it was difficult to distinguish more than
% two notes. The 0 Hz term is not audible. Please see the answer in part (b).

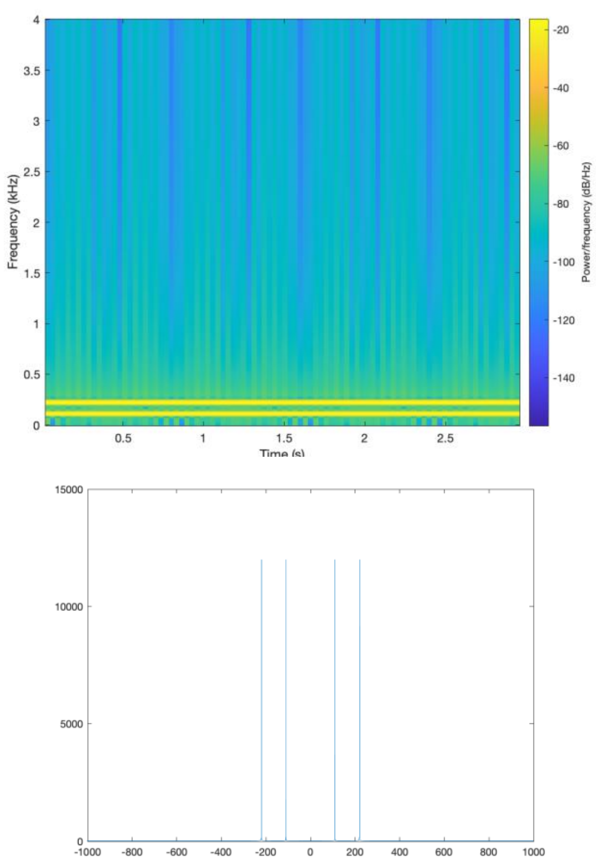
```
soundsc(y, fs);
pause(tmax+1);
```

% (f) Plot the spectrum of the sampled version of x(t). *Principal frequencies*
% are 0, 110, 220, 330, and 440 Hz. Principal frequencies at 0, 330, 440 Hz aren't
% in x(t) and are called intermodulation distortion caused by the squaring system.
% **Using the spectrogram**. *See below for the plot*.

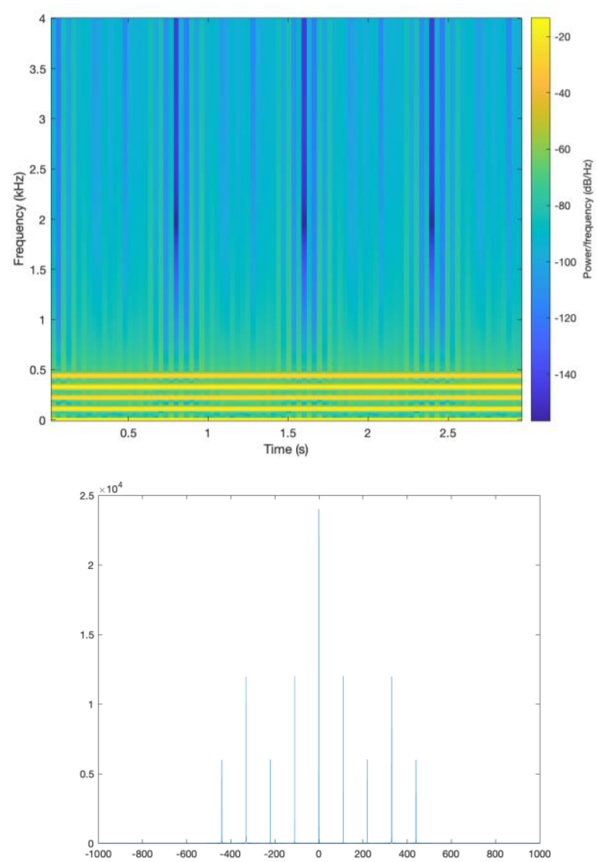
```
figure;
spectrogram(y, 512, 256, 512, fs, 'yaxis');
```

% **Using the fast Fourier transform approach from mini-project #1**. *See below*.

```
fourierSeriesCoeffs = fft(y);
N = length(y);
freqResolution = fs / N;
ff = (-fs/2) : freqResolution : (fs/2)-freqResolution;
figure;
plot(ff, abs(fftshift(fourierSeriesCoeffs)));
xlabel('f');
xlim( [-1000, 1000] );
```



Spectrogram (*above*)
and spectrum (*below*)
for sampled *x*(*t*)



Spectrogram (*above*)
and spectrum (*below*)
for sampled *y*(*t*)

Amplitude modulation forms the basis for AM/FM radio, Wi-Fi, cellular, cable modems, and other communication systems. Amplitude modulation can create audio effects. AM radio adds a constant offset to the audio signal whereas other types of amplitude modulation do not.

Problem 1.3. *AM Radio.* 30 points**.**

SPFirst Sec. 3-3, 3-6, 4-1 and 4-2

Lecture slides 3-4 to 3-14, 5-4 to 5-6, 5-9 to 5-14, and 6-3 to 6-9

Homework Prob. 2.1, 2.2 & 3.2

Tune-Up Tuesdays 1-3

Each AM radio station operates at a fixed broadcast frequency f_c in the range 530 kHz to 1700 kHz.

Amplitude Modulation. An AM radio transmitter converts an audio signal $x(t)$ to a radio signal $s(t)$ via

$$s(t) = (\, x(t) + A\,) \cos(2\,\pi\, f_c\, t)$$

where A is a constant chosen to be greater than the maximum value of $|\, x(t)\,|$.

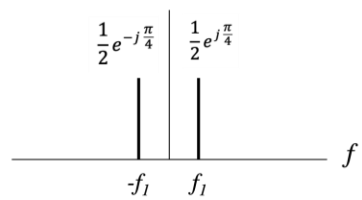
For this problem, use $f_c = 1300$ kHz and $x(t) = \cos\left(2\pi\, f_1\, t + \frac{\pi}{4}\right)$ where $f_1 = 700$ Hz.

(a) Draw the spectrum for $x(t)$. *6 points.*

$$\boldsymbol{x(t) = \cos\left(2\pi\, f_1\, t + \frac{\pi}{4}\right)}$$

$$\boldsymbol{x(t) = \frac{1}{2} e^{j\,2\pi\, f_1\, t + \frac{\pi}{4}} + \frac{1}{2} e^{-j\,2\pi\, f_1\, t - \frac{\pi}{4}}}$$

$$\boldsymbol{x(t) = \frac{1}{2} e^{j\frac{\pi}{4}}\, e^{j\,2\pi\, f_1\, t} + \frac{1}{2} e^{-j\frac{\pi}{4}}\, e^{-j\,2\pi\, f_1\, t}}$$



(b) Draw the spectrum for $s(t)$. *6 points.*

$$\boldsymbol{s(t) = (x(t) + A) \cos(2\pi\, f_c\, t)}$$

$$\boldsymbol{= x(t)\cos(2\pi\, f_c\, t) + A\cos(2\pi\, f_c\, t)}$$

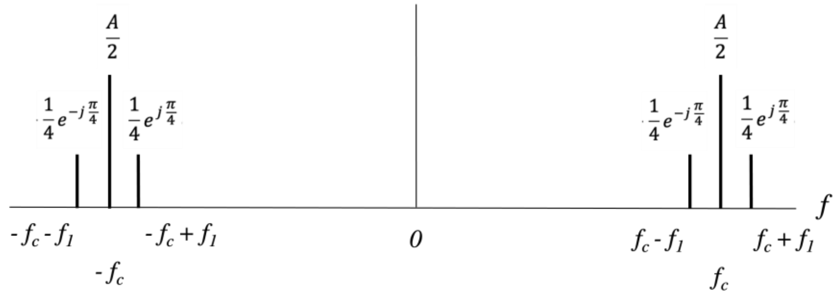
Right term

$$\boldsymbol{A\cos(2\pi\, f_c\, t) = \frac{A}{2} e^{-j\,2\pi\, f_c\, t} + \frac{A}{2} e^{j\,2\pi\, f_c\, t}}$$

Left term

$$\boldsymbol{x(t)\cos(2\pi\, f_c\, t) = \cos\left(2\pi\, f_1\, t + \frac{\pi}{4}\right)\cos(2\pi\, f_c\, t)}$$

$$\boldsymbol{= \frac{1}{4}\left(e^{j(2\pi(f_1+f_c)t)} e^{j\left(\frac{\pi}{4}\right)} + e^{-j(2\pi(f_c-f_1)t)} e^{j\left(\frac{\pi}{4}\right)} + e^{j(2\pi(f_c-f_1)t)} e^{-j\left(\frac{\pi}{4}\right)} + e^{-j(2\pi(f_1+f_c)t)} e^{-j\left(\frac{\pi}{4}\right)}\right)}$$



Amplitude Demodulation. We will use undersampling in the demodulation process to convert the received AM radio signal $s(t)$ to an audio signal $\hat{x}(t)$.

(c) Sample the AM radio signal $s(t)$ at a sampling rate of $f_s = 650$ kHz to create the signal $s[n]$. Give a formula for $s[n]$. *6 points.*

$$\boldsymbol{s(t) = \frac{1}{2}\cos\left(2\pi\,(f_c - f_1)\, t - \frac{\pi}{4}\right) + A\cos(2\pi\, f_c\, t) + \frac{1}{2}\cos\left(2\pi\,(f_c + f_1)\, t + \frac{\pi}{4}\right)}$$

After sampling, the three positive frequency values become

$$\boldsymbol{\hat{\omega}_{f_c - f_1} = 2\pi \frac{f_c - f_1}{fs} \text{ and } \hat{\omega}_c = 2\pi \frac{f_c}{fs} = 2\pi \frac{1300 \text{ kHz}}{650 \text{ kHz}} = 4\pi \text{ and } \hat{\omega}_{f_c + f_1} = 2\pi \frac{f_c + f_1}{fs}}$$

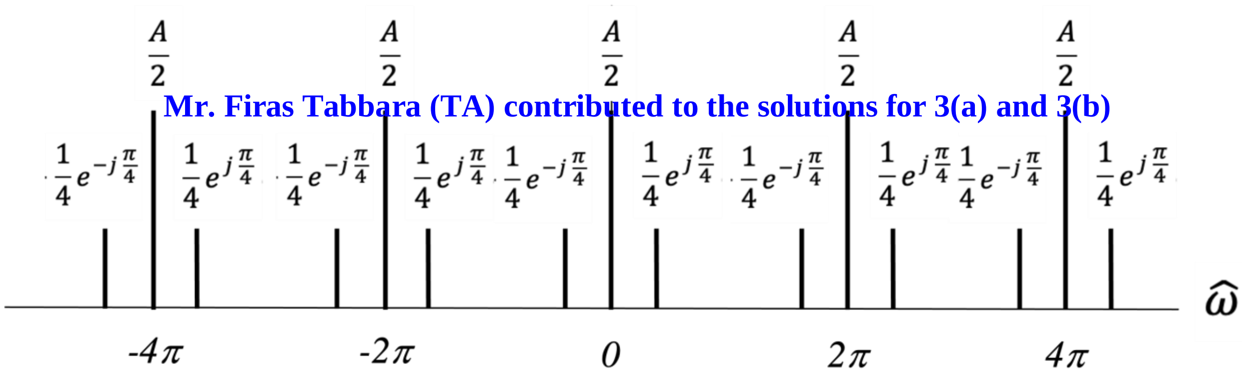
$$\boldsymbol{s[n] = s(nT_s) = \frac{1}{2}\cos\left(\hat{\omega}_{f_c - f_1}\, n - \frac{\pi}{4}\right) + A\cos(\hat{\omega}_c\, n) + \frac{1}{2}\cos\left(\hat{\omega}_{f_c + f_1} n + \frac{\pi}{4}\right)}$$

(d) Draw the spectrum for $s[n]$ for discrete-time frequencies $-5\pi < \hat{\omega} \leq 5\pi$. *6 points.*

The frequency 4π will alias to 2π, 0, -2π, -4π, etc., as well as 6π, 8π, etc.

The frequency -4π will alias to -2π, 0, 2π, 4π, etc., as well as -6π, −8π, etc.

Mr. Firas Tabbara (TA) contributed to the solutions for 3(a) and 3(b)



(e) What continuous-time frequencies would be present when reconstructing a continuous-time signal from $s[n]$? Please include negative, zero, and positive frequencies, if present. *6 points.*

Discrete-time frequencies in rad/sample used in reconstruction are $-\pi < \hat{\omega} \leq \pi$

Continuous-time frequencies in Hz used in reconstruction are $-\frac{1}{2}f_s < f \leq \frac{1}{2}f_s$ **which is** $-325 \text{ kHz} < f \leq 325 \text{ kHz}$ **because** $f_s = 650 \text{ kHz}$.

Frequencies present when reconstructing the continuous-time signal from *s*[*n*] are -700 Hz, 0, and 700 Hz. This is the same frequencies that are present in $x(t) + A$.

When computing transforms that involve integration or summation, we will not compute the answer from the mathematical definition every time. Instead, we will build up a table of transform results for commonly used signals, and then use the table and transform properties to transform other signals.

SPFirst Sec. 3-3 to 3-5 | *Lecture slides 3-7 to 3-14* | *Homework Prob. 2.4 & 3.1*

Problem 1.4. *Fourier Series Properties.* 24 points.

The continuous-time Fourier series has several properties.

For example, if $y(t) = A\,x(t)$ and $x(t)$ is periodic with fundamental period f_0 and Fourier series coefficients a_k, then the Fourier series coefficients b_k for $y(t)$ can be found using $b_k = A\,a_k$:

$$y(t) = A\,x(t) = A \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{\infty} A\, a_k\, e^{j2\pi(kf_0)t}$$

For the following expressions, derive the relationship between the Fourier series coefficients b_k for $y(t)$ and the Fourier series coefficients a_k for $x(t)$ where

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

(a) $y(t) = x(t-T)$. *6 points.* **Time Shift/Delay Property for the Fourier Series.**

$$\boldsymbol{y(t) = x(t-T) = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(kf_0)(t-T)} = \sum_{k=-\infty}^{\infty} a_k\, e^{-j2\pi(kf_0)T}\, e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{+\infty} b_k\, e^{j2\pi(kf_0)t}}$$

Here, $b_k = a_k\, e^{-j2\pi(kf_0)T}$

(b) $y(t) = x(-t)$. *9 points.* **Time Reversal Property for the Fourier Series.**

$$\boldsymbol{y(t) = x(-t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(kf_0)(-t)} = \sum_{k=-\infty}^{\infty} a_k\, e^{j2\pi(-kf_0)t}}$$

Let $m = -k$. Limits of summation change: as $k \to \infty, m \to -\infty$ and as $k \to -\infty, m \to \infty$.

$$\boldsymbol{y(t) = x(-t) = \sum_{m=\infty}^{-\infty} a_{-m}\, e^{j2\pi(mf_0)t} = \sum_{m=-\infty}^{\infty} a_{-m}\, e^{j2\pi(mf_0)t}}$$

because the order in which one sums over the same values of the summation variable does not affect the result.

Here, $b_k = a_{-k}$

(c) $y(t) = \cos(2\,\pi\, f_0\, t)\, x(t)$. *9 points.* **Modulation Property for the Fourier Series.**

$$\boldsymbol{b_k = \frac{1}{T_0}\int_0^{T_0} y(t)\, e^{-j2\pi(kf_0)t}\, dt = \frac{1}{T_0}\int_0^{T_0} \cos(2\,\pi\, f_0\, t)\, x(t)\, e^{-j2\pi(kf_0)t}\, dt}$$

$$\boldsymbol{b_k = \frac{1}{T_0}\int_0^{T_0} \left(\frac{1}{2}\, e^{-j2\pi f_0 t} + \frac{1}{2}\, e^{j2\pi f_0 t}\right) x(t)\, e^{-j2\pi(kf_0)t}\, dt}$$

$$\boldsymbol{b_k = \frac{1}{2}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi f_0 t}\, e^{-j2\pi(kf_0)t}\, dt\right) + \frac{1}{2}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{j2\pi f_0 t}\, e^{-j2\pi(kf_0)t}\, dt\right)}$$

$$\boldsymbol{b_k = \frac{1}{2}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi((k-1)f_0)t}\, dt\right) + \frac{1}{2}\left(\frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-j2\pi((k+1)f_0)t}\, dt\right)}$$

Here, $b_k = \frac{1}{2} a_{k-1} + \frac{1}{2} a_{k+1}$

In part (c), multiplying a signal $x(t)$ by $\cos(2\,\pi\, f_0\, t)$ causes each frequency component in $x(t)$ to shift left by f_0 and be scaled in amplitude by ½ and each frequency component in $x(t)$ to shift right by f_0 and be scaled in amplitude by ½ . This is creating a beat frequency between f_0 and every frequency component of $x(t)$.